

# Finite $\mathcal{W}$ -algebras and intermediate statistics

F. Barbarin<sup>1</sup>, E. Ragoucy<sup>1</sup>, P. Sorba<sup>1,2</sup>

*Laboratoire de Physique Théorique ENSLAPP \**

## Abstract

New realizations of finite  $\mathcal{W}$ -algebras are constructed by relaxing the usual constraint conditions. Then finite  $\mathcal{W}$ -algebras are recognized in the Heisenberg quantization recently proposed by Leinaas and Myrheim, for a system of two identical particles in  $d$  dimensions. As the anyonic parameter is directly associated to the  $\mathcal{W}$ -algebra involved in the  $d = 1$  case, it is natural to consider that the  $\mathcal{W}$ -algebra framework is well adapted for a possible generalization of the anyon statistics.

*in memory of our friend and colleague Tanguy Altherr*

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\*URA 14-36 du CNRS, associée à l'E.N.S. de Lyon, et au L.A.P.P. (IN2P3-CNRS) d'Annecy-le-Vieux.

<sup>1</sup> Groupe d'Annecy, LAPP, Chemin de Bellevue, BP 110, F - 74941 Annecy-le-Vieux Cedex, France

<sup>2</sup> Groupe de Lyon, ENS, 46 allée d'Italie, F-69364 Lyon Cedex 07, France

# 1 Introduction

Finite  $\mathcal{W}$ -algebras have been introduced [1, 2] by studying symplectic reductions of finite dimensional simple Lie algebras in complete analogy with usual  $\mathcal{W}$ -algebras constructed as reductions of affine Lie algebras. It is reasonable to think, with the authors of [2], that a good understanding of the finite case can help to draw informations on the infinite dimensional one. In the same time, one must not be surprised if some of such finitely generated but non linear algebras already appeared in theoretical physics. That is the case of the finite  $\mathcal{W}$ -algebra obtained from the reduction of the  $sp(4, \mathbb{R})$  model associated with the  $sl(2, \mathbb{R})$  embedding of Dynkin index 2 (i.e. built on the short root of  $sp(4, \mathbb{R})$ ). The corresponding  $\mathcal{W}$ -algebra is four dimensional, and can be seen as a "deformed" version of  $gl(2, \mathbb{R})$  with a cubic term in the r.h.s. of a commutation relation (see below). As remarked in [3], this algebra has already been considered by P.W. Higgs as an algebra of conserved quantities for a Coulombic central force problem in a space of constant curvature [4].

We have recognized the same finite  $\mathcal{W}$ -algebra in a completely different framework, namely intermediate statistics. Indeed, in [5], J.M. Leinaas and J. Myrheim have considered the Heisenberg quantization of a system of two identical particles in one and two dimensions. The one-dimensional system can be formally related to a system of two identical vortices in a thin, incompressible superfluid film, the two spatial coordinates of the vortex center acting as canonically conjugate quantities. Such a model has been proposed as anyon candidate. The authors have remarked that one-dimensional Heisenberg quantization is closely related to Schrödinger quantization in two dimensions, group theory playing an important rôle in the former approach. Indeed, in the Heisenberg framework, intermediate statistics [11] are clearly characterized by a continuous parameter, itself related to the Casimir eigenvalue of the  $sp(2, \mathbb{R})$  observable algebra. The boson and fermion cases correspond to limiting values of this parameter.

The situation is more complicated in the two-dimensional Heisenberg quantization, from the algebraic point of view as well as from the physical interpretation. The algebra of interest is now  $sp(4, \mathbb{R})$  and more than one extra parameter are necessary to characterize the different configurations. Although many points still need to be clarified for a generalization of the anyon notion, one remarks that in the two cases considered above, the physical interpretation is brought by a finite  $\mathcal{W}$ -algebra. In the one dimensional case, it is just the  $sl(2, \mathbb{R})$  Casimir operator, while in the two-dimensional one, it is exactly the finite  $\mathcal{W}$ -algebra already described in [3] and [4]. Reconsidering the problem in the  $\mathcal{W}$ -algebra framework can help a lot for setting more rapidly and systematically the algebraic formalism used in [5].

It is one of the purpose of this paper to present this approach, which might be of some use to go further in the attempt of [5] to generalize intermediate statistics.

Moreover, the determination of the finite  $\mathcal{W}$ -algebras in the above examples has to be done without putting to constant the current components relative to the constraints in the usual Hamiltonian reduction. In other words, these components are left free, although they still determine the transformations under which the  $W$  generators will be invariant. Only the case where such components are commuting (or in other words the gauge group is Abelian) will be considered here. Then, as will be shown on examples, the  $W$  generators depend explicitly on all the current components, but their polynomiality property (e.g. in the highest weight gauge) is lost. This property can be restored by enlarging the  $\mathcal{W}$ -algebra

to a bigger algebra containing also the component formerly related to the constraints (i.e. dual to the group transformations).

Such a construction of finite  $\mathcal{W}$ -algebras by keeping free the components associated to the constraints is the second point that we would like to emphasize in this paper. Of course, we hope that such a feature of finite  $\mathcal{W}$ -algebras could also be extended to the affine case, thus proving, following the previous assertions, that finite  $\mathcal{W}$ -algebras constitute a suitable laboratory for more elaborated structures.

The paper is constituted by two sections. In the first one, we realize finite  $\mathcal{W}$  generators as invariant quantities under Abelian subgroups  $G_+$  of a simple group  $G$ . Some of their properties are discussed, in particular the correspondence, after quantization, between  $\mathcal{W}$  and the commutant of the Abelian subalgebra  $\mathcal{G}_-$ . These technics appear well adapted to study the representations of the Lie algebra  $sp(2d, \mathbb{R})$  to which reduces, following [5], the Heisenberg quantization for a system of two identical particles in  $d$  dimensions. As in [5], the  $d = 1$  and  $d = 2$  cases are considered in detail, with emphasis each time on the rôle of the associated finite  $\mathcal{W}$ -algebra, while general features for  $d > 2$  are rapidly mentioned. Finally we suggest an alternative to the conclusion of [5] in order to recognize, through the  $\mathcal{W}$ -algebra in the  $d = 2$  case, the anyonic parameter well established for  $d = 1$ .

## 2 New realizations of finite $\mathcal{W}$ -algebras

We recall in the first paragraph some features of the construction of finite  $\mathcal{W}$ -algebras from Lie algebras. Then, we present new realizations for these  $\mathcal{W}$ -algebras. In opposition to the usual Hamiltonian reduction, no constraints are necessary, so that these realizations use all the generators of the Lie algebra that lies behind the finite  $\mathcal{W}$ -algebra. Finally, we quantize the previous classical approach.

### 2.1 The traditional Hamiltonian reduction

As developed in [1, 2], Hamiltonian reduction of Poisson structures on simple Lie algebras can be performed to construct finite  $\mathcal{W}$ -algebras, in complete analogy with the affine case, the only difference being the absence of the space variable.

We start with a simple, real, connected and maximally non compact Lie group  $G$ , with Lie algebra  $\mathcal{G}$ . Let  $t^a$  be the basis of  $\mathcal{G}$ , and  $J_a$  the dual basis in  $\mathcal{G}^*$ :

$$[t^a, t^b] = f_c^{ab} t^c \quad J_a(t^b) = \delta_a^b \quad (2.1)$$

We introduce the metric on  $\mathcal{G}$  in a representation  $R$ :

$$\eta^{ab} = \langle t^a, t^b \rangle = tr_R(t^a t^b) \quad \text{and} \quad \eta_{ab} \eta^{bc} = \delta_a^c \quad (2.2)$$

One can define on  $\mathcal{G}^*$  a Poisson-Kirillov structure which mimicks the commutators (2.1):

$$\{J_a, J_b\} = f_{ab}^c J_c \quad \text{with} \quad f_{ab}^c = \eta_{ad} \eta_{be} \eta^{cg} f_g^{de} \quad (2.3)$$

Now, as usual [6, 7], we introduce a gradation on  $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$  relative to an  $sl(2, \mathbb{R})$  embedding w.r.t. which the reduction will be performed. We will call  $G_{0,+,-}$  the subgroups

associated to  $\mathcal{G}_{0,+,-}$ . In order not to overload the presentation, let us limit our discussion to the integral grading case. The half-integral gradations need some extra precautions: we will add a few remarks on it in the next paragraph. As a notation we use the indices  $\alpha, \beta, \dots$  for negative grades,  $i, j, k, \dots$  for the zero grades, and  $\bar{\alpha}, \bar{\beta}, \dots$  for the positive ones. The corresponding sets will be denoted by  $I_{\pm,0}$ . We keep  $a, b, c, \dots$  for the general indexation, and  $t^{0,+,-}$  are the generators of the  $sl(2, \mathbb{R})$  under consideration.

Now, we introduce on  $\mathcal{G}^*$  first class constraints relative to the  $\mathcal{G}_-^*$  part:

$$J_\alpha - \chi_\alpha = 0, \quad \forall \alpha \in I_- \quad (2.4)$$

where  $\chi_\alpha$  is a constant, which is zero except when  $J_\alpha = J_-$ , the negative  $sl(2, \mathbb{R})$  root generator. We take  $\chi_- = 1$  for simplicity. The (first class) constraints weakly commute among themselves:

$$\{J_\alpha - \chi_\alpha, J_\beta - \chi_\beta\}_{Const} = 0 \quad (2.5)$$

where the index *Const* indicates that one has to apply the constraints after computation of the Poisson Bracket (PB). Following Dirac prescription, these first class constraints generate a gauge invariance on the  $J_a$ 's:

$$J_a \rightarrow \exp(c^\alpha \{J_\alpha, \cdot\}_{Const})(J_a) = J_a + c^\alpha \{J_\alpha, J_a\}_{Const} + \frac{1}{2} c^\alpha c^\beta \{J_\beta, \{J_\alpha, J_a\}\}_{Const} + \dots \quad (2.6)$$

where the  $c^\alpha$  are the gauge transformation parameters.

It is useful to keep in mind that these gauge transformations can be performed either through commutation relation on  $\mathcal{G}$ , or via PB in  $\mathcal{G}^*$ . Indeed, let us introduce the usual constrained matrix

$$J = t^- + J_i t^i + J_{\bar{\alpha}} t^{\bar{\alpha}} \quad (2.7)$$

and look at the gauge transformation (2.6):

$$J \rightarrow J^g = \exp(c^\alpha \{J_\alpha, \cdot\}_{Const})(J) \quad (2.8)$$

Developing  $J^g$  with the help of the gradation and the use of the constraints, we can, thanks to the relations (2.1-2.3), rewrite  $J^g$  as

$$J^g = \exp([c_{\bar{\alpha}} t^{\bar{\alpha}}, \cdot])(J) = g^{-1} J g \quad (2.9)$$

where we have introduced the group element  $g = \exp(c_{\bar{\alpha}} t^{\bar{\alpha}})$  and the parameters  $c_{\bar{\alpha}} = \eta_{\bar{\alpha}\alpha} c^\alpha$ . Thus, the gauge transformations can be seen as conjugations on  $\mathcal{G}$  by elements of the subgroup  $G_+$ . Note that at the affine level, the above discussion is still valid but, because of the central term in the PB (2.3), one finds the (usual) coadjoint transformation instead of conjugation. Here, the differential term being identically zero, the conjugation and the coadjoint transformation are identical.

Finally, one fixes the gauge (2.9) by demanding the transformed current  $J^g$  to be of the form (highest weight gauge)

$$J^g = t^- + W_s \ell^s \quad \text{with} \quad [t^+, \ell^s] = 0 ; [t^0, \ell^s] = s \ell^s \quad (2.10)$$

The  $W_s$ 's are in the enveloping algebra of  $\mathcal{G}^*$ , and they are gauge invariant. They form a basis of the finite  $\mathcal{W}$ -algebra. Of course, by construction, the  $W$  generators, which are the gauge invariant polynomials in the  $J_i$ 's and  $J_{\bar{\alpha}}$ 's will have *weakly* vanishing PB with the constraints.

## 2.2 Relaxing the constraints

Instead of using the constrained current (2.7), one can perform the coadjoint transformations on the complete matrix  $J_{tot} = J_a t^a$ :

$$J_{tot}^g = g^{-1} J_{tot} g \quad \text{with} \quad g \in G_+ \quad (2.11)$$

Then, developing  $J_{tot}^g$  using the same rules as in the previous section, we get

$$J_{tot}^g = \exp(c^\alpha \{J_\alpha, \cdot\})(J_{tot}) \quad (2.12)$$

where now the PB are computed *without* any use of what were the constraints. Thus, if one finds (as we will show below) quantities which are invariant under the coadjoint transformations, these objects will have *strongly* vanishing PB with the generators  $J_\alpha$ .

Note that, although the transformations we are looking at have the same form as the gauge transformations of the previous section, they are *not* gauge transformations, since they are not associated to constraints. In other words, one cannot construct an action associated to the transformations (2.11), while for the previous section, the action

$$S(g, A_+, A_-) = \text{tr} \int dx_0 \frac{dg}{dx_0} \frac{d}{dx_0} g^{-1} + \text{tr} \int dx_0 A_+ (g^{-1} \frac{dg}{dx_0} - t^-) + (\frac{dg}{dx_0} g^{-1} - t^+) A_- + g^{-1} A_+ g A_- \quad (2.13)$$

is really invariant under the gauge transformations

$$g(x_0) \rightarrow g_+(x_0) g(x_0) g_-(x_0) \quad (2.14)$$

Thus, the construction we present is strictly algebraic, but we will see in the next section that the technics can be applied to physical problems.

We will limit our study to the case where the constraints (2.5) have vanishing PB, that is to say when the conjugation group  $G_+$  is Abelian.

Note that, if  $G_+$  is Abelian, the  $\mathcal{G}$ -gradation reduces to

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1} \quad (2.15)$$

Indeed, (2.15) insures that  $\mathcal{G}_{-1}$  is Abelian. Conversely, as we are considering gradation w.r.t. an  $sl(2, \mathbb{R})$  Cartan generator, the simple roots have grades  $0 \leq h \leq 1$  only. Thus, a grade -2 generator (if it exists) is not a simple root generator, and must be obtained as the commutation of two simple root generators of grade -1, so that  $\mathcal{G}_{-1}$  would not be Abelian. An analogous reasoning for half gradations leads to

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_{+1} \quad (2.16)$$

Let us add that in that case, using the halving technics [7, 6], one can split  $\mathcal{G}_{-\frac{1}{2}}$  in such a way that the second class constraints become first class. In most of the cases, an half-integral gradation can be redefined into an integral one by using an  $sl(2, \mathbb{R})$ -commuting  $U(1)$  factor [8].

Using [9], it is easy to see that the gradations of types (2.15-2.16) correspond to a reduction w.r.t.:

(i) the diagonal  $sl(2, \mathbb{R})$  in  $p$  commuting  $sl(2, \mathbb{R})_1$  (the subscript is the Dynkin index,  $1 \leq p \leq [\frac{N}{2}]$ ) for  $\mathcal{G} = sl(N)$ ;

- (ii)  $sl(2, \mathbb{R})_1$  or the diagonal  $sl(2, \mathbb{R})$  in  $p$  commuting  $sl(2, \mathbb{R})_2$  ( $1 \leq p \leq d$ ) for  $\mathcal{G} = sp(2d)$ ;
- (iii) the diagonal  $sl(2, \mathbb{R})$  in  $p$   $sl(2, \mathbb{R})_1$  ( $1 \leq p \leq [\frac{N}{2}]$ ) for  $\mathcal{G} = so(N)$ .

Now, coming back to the transformations (2.11), it is easy to see that, because  $G_+$  is Abelian, the negative grade generators in  $J_{tot}^g$  are not affected by  $g$ :

$$J_{tot}^g = J_\alpha t^\alpha + K_{\geq 0} \quad (2.17)$$

with  $K_{\geq 0}$  decomposing on the positive grades  $\mathcal{G}_0 \oplus \mathcal{G}_+$  only:  $K_{\geq 0} = K_i t^i + K_{\bar{\alpha}} t^{\bar{\alpha}}$ . Then, it is natural to define

$$\tilde{t}^- = J_\alpha t^\alpha \quad (2.18)$$

This generator being nilpotent, it can be included in an  $sl(2, \mathbb{R})$  subalgebra<sup>1</sup>  $\{\tilde{t}_\pm, \tilde{t}_0\}$ . Once the other generators  $\tilde{t}_+, \tilde{t}_0$  have been identified, we can take as symmetry fixing a highest weight form:

$$J_{tot}^g = J_\alpha t^\alpha + \tilde{W}_s \tilde{\ell}^s \quad \text{with} \quad [\tilde{t}^+, \tilde{\ell}^s] = 0 \quad (2.19)$$

where the index  $s \in I_{hw}$  labels the highest weight generators. Then, we have the property:

**Proposition 1** *The set  $\mathcal{S} = \{\tilde{W}_s, J_\alpha; s \in I_{hw}, \alpha \in I_-\}$  defines an enlarged finite  $\mathcal{W}$ -algebra called  $\mathcal{W}_\mathcal{S}$ , closing (w.r.t. PB) on rational functions of the form  $P(\tilde{W}_s, J_\alpha)/Q(J_\alpha)$ , and realized by quotients of polynomials in the  $J_a$ 's over polynomials only in the  $J_\alpha$ 's.*

*Moreover, the subset  $\mathcal{G}_-^* = \{J_\alpha; \alpha \in I_-\}$  is in the center of the  $\mathcal{W}_\mathcal{S}$  algebra.*

*Proof:*

It is clear, using the same arguments as for the usual highest weight gauge [7], that the highest weight form determines uniquely the symmetry generator  $g$ . Thus,  $J_{tot}^g$  will provide a complete set of symmetry invariant generators. One has to be careful that these generators will not only be the  $\tilde{W}_s$  generators but also the  $J_\alpha$ 's. Moreover, from the form of  $\tilde{t}^-$  and the polynomiality of the usual highest weight gauge, one deduces the  $\tilde{W}_s$  generators are quotients of polynomials in the  $J_a$ 's over polynomials in the  $J_\alpha$ 's only. Note that since the  $\tilde{W}_s$  are fractions (and no more polynomials), the set  $\mathcal{S}$  generates a complete set of invariant generators through rational functions  $P(J_a)/Q(J_\alpha)$  instead of polynomials. However, we remark that any invariant polynomial  $P(J_a)$  can be rewritten as a polynomial  $Q(\tilde{W}_s, J_\alpha)$ . Indeed, starting from an invariant polynomial  $P(J_a)$ , let us perform the symmetry transformation  $g$  that leads to the highest weight form. Then, from the invariance of  $P$  and the highest weight form, we get:

$$P(J_a) = P(J_a)^g = P(J_a^g) = P(\tilde{W}_s, J_\alpha, 0) \equiv Q(\tilde{W}_s, J_\alpha) \quad (2.20)$$

We note that the PB of two  $\tilde{W}_s$  generators reads

$$\{\tilde{W}_s', \tilde{W}_s''\} = \frac{P(\tilde{W}_s, J_\alpha)}{Q(J_\alpha)} \quad (2.21)$$

We conjecture that we can get rid of the denominator  $Q(J_\alpha)$  and get a "reduced" polynomial  $P_r(\tilde{W}_s, J_\alpha)$ . If this conjecture is true, then  $\mathcal{W}_\mathcal{S}$  will close polynomially in the  $\tilde{W}_s$ 's and  $J_\alpha$ 's. In the section 3.2, we show explicitly an example where the conjecture is verified.

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<sup>1</sup>In our construction, we discard the case where  $\tilde{t}_- = 0$ .

Finally, because of the invariance, any element of  $\mathcal{S}$  will commute with the  $J_\alpha$ 's, so that  $\mathcal{G}_-^*$  is in the center of  $\mathcal{W}_\mathcal{S}$ .

It should be obvious that the original  $\mathcal{W}$ -algebra (obtained by Hamiltonian reduction) can be deduced from the enlarged  $\mathcal{W}$ -algebra just by applying the constraints that change  $\tilde{t}_+$  into  $t_+$ . Moreover, the new generators being in the center, one can consider the quotient by  $\mathcal{G}_-^*$ . As we are looking at  $\mathcal{W}$  algebras, the definition of the quotient has to be slightly modified:

**Definition 1** *Let  $\mathcal{W}_\mathcal{S}$  be a  $\mathcal{W}$ -algebra and  $\mathcal{S}_0$  a subset of the center of  $\mathcal{W}_\mathcal{S}$ , one defines the quotient  $\mathcal{W}_\mathcal{S}/\mathcal{S}_0$  through the equivalence relation*

$$W \equiv W' \Leftrightarrow W = F.W' + G \quad (2.22)$$

where  $F, G$  are smooth functions of the elements of  $\mathcal{S}_0$  such that  $F \neq 0$ .

Starting from this definition, one can consider the PB  $\{[W], [W']\}' \doteq \{[W, W']\}$ , where  $[W]$  is an element of  $\mathcal{W}_\mathcal{S}/\mathcal{G}_-^*$ . In this PB one can set formally the  $J_\alpha$ 's to the constraints  $\chi_\alpha$  of section 2.1, since they are no more present in the quotient. But on the hyperplan  $J_\alpha = \chi_\alpha$ , the  $\mathcal{W}_\mathcal{S}$  is the  $\mathcal{W}$  algebra of the previous section. Thus, we have

**Proposition 2** *The quotient  $\mathcal{W}_\mathcal{S}/\mathcal{G}_-^*$  is the finite  $\mathcal{W}$ -algebra associated to the gauge group  $G_+$ .*

From proposition 2, we deduce

**Corollary 1** *Starting from the enlarged  $\mathcal{W}$ -algebra associated to the symmetry group  $G_+$ , there is a consistent way of "renormalizing" the generators  $\tilde{W}_s$  by smooth functions of the  $J_\alpha$ 's in such a way that the renormalized generators  $\tilde{W}_s$  form the  $\mathcal{W}$ -algebra associated to the group  $G_+$ .*

## 2.3 Quantization

The previous realizations use generators in the dual space  $\mathcal{G}^*$ . In the case of (usual) Hamiltonian reduction, it means that we are working at the classical level. However, it is often useful (as we will see) to have a realization of the  $\mathcal{W}$ -algebra on  $\mathcal{G}$  itself, thereby quantizing (in the Hamiltonian framework case) the system. For finite  $\mathcal{W}$ -algebras, the work is easy since there always exists an (algebra) isomorphism between  $\mathcal{G}$  and its dual. After reminding this isomorphism, we will extend it to the enveloping algebra  $\mathcal{U}^*$  of  $\mathcal{G}^*$ , and then apply it to the realization of finite  $\mathcal{W}$ -algebras.

Let  $i$  be the isomorphism between  $\mathcal{G}^*$  and  $\mathcal{G}$ :

$$i(J_a) = t_a = \eta_{ab} t^b \quad \text{with} \quad t_a(t^b) = \langle t_a, t^b \rangle$$

where  $\eta_{ab}$  is the inverse matrix of the metric  $\eta^{ab}$ . This isomorphism is a Lie algebra isomorphism that sends the PB into the commutator:

$$[i(J_a), i(J_b)] = \eta_{ad}\eta_{be} [t^d, t^e] = f_{ab}{}^c i(J_c) = i(\{J_a, J_b\})$$

It allows to relate the coadjoint and the adjoint actions. Indeed, the coadjoint action  $Ad^*$  is defined by

$$[Ad^*(g)(J_a)](X) = J_a(Ad(g^{-1})(X)) \quad \forall X \in \mathcal{G} \quad \text{with} \quad Ad(g)(X) = gXg^{-1} \quad g \in G$$

Using  $i$  and the group-invariance of  $\langle ., . \rangle$  one gets

$$\langle t_a, g^{-1}Xg \rangle = \langle gt_ag^{-1}, X \rangle = \langle Ad(g)(t_a), X \rangle$$

so that

$$i \circ Ad^* = Ad \circ i$$

Now, we extend  $i$  to the enveloping algebra  $\mathcal{U}(\mathcal{G}^*) \equiv \mathcal{U}^*$  with the rule

$$i(J_{a1} \cdot J_{a2} \cdots J_{an}) = S(t_{a1}, t_{a2}, \dots, t_{an}) \quad \forall n \quad (2.23)$$

where  $S(., ., \dots, .)$  stands for the *symmetrized* product of the generators  $t_a$ .  $S$  is normalized by  $S(X, X, \dots, X) = X^n$ .

Note that  $i$  is a vectorspace isomorphism between  $\mathcal{U}^*$  and the set  $\mathcal{U}_{Sym}$  of totally symmetric polynomials in the  $t_a$ 's. This set is a algebra (w.r.t. the commutator), but  $i$  is no more an algebra isomorphism. However,  $i$  satisfies the fundamental property

$$i(\{J_a, P\}) = [t_a, i(P)] \quad \forall J_a \in \mathcal{G}^*, \forall P \in \mathcal{U}^* \quad (2.24)$$

Then, this implies that the symmetry transformations are "preserved" by  $i$ .

Now, starting with a *polynomial*  $W \in \mathcal{W}_S$ , one can construct the operator  $\overline{W} = i(W) \in \mathcal{U}_{Sym}$ . Since  $W$  is invariant under the group transformations (2.12), we deduce that  $\overline{W}$  will be invariant under the transformations

$$\overline{W} \rightarrow \exp(c^\alpha [t_\alpha, .])(\overline{W}) \quad (2.25)$$

Of course, the above construction can be done in  $\mathcal{U}_{Sym}$  *a priori* only for polynomials. However, as  $\mathcal{G}_-$  is in the center of  $\mathcal{W}_S$ , if one considers the subset  $i(\mathcal{W}_S)$  in  $\mathcal{U}_{Sym}$ , quotients by polynomials  $P(J_\alpha)$  will be allowed. Thus,  $i(W)$  will be well-defined in  $i(\mathcal{W}_S)$  for any element in  $\mathcal{W}_S$ .

*In other words, if  $W$  has vanishing PB with the  $J_\alpha$ 's, then  $\overline{W}$  will commutes with the image of these generators, that is the  $t_\alpha$ 's.*

*The set of  $\overline{W}$  fields being a complete set of invariant under the transformations (2.25), it is clear that it will form a quantized enlarged  $\mathcal{W}$ -algebra.*

Let us remark that although  $i$  is an algebra isomorphism between  $\mathcal{G}^*$  and  $\mathcal{G}$ , it is not an algebra isomorphism between  $\mathcal{U}^*$  and  $\mathcal{U}_{Sym}$ . Thus, one has to be careful that the commutation relations between  $\overline{W}$  generators may be different from the PB between  $W$  fields: an example of the difference between the classical and quantized  $\mathcal{W}$ -algebras can be found in [10].

### 3 Intermediate statistics and finite $\mathcal{W}$ -algebras

As an application of the results of section 2, we consider the Heisenberg quantization for systems of two identical particles as developed in [5]. Although the one dimensional case is algebraically simple, it deserves to be presented, first as a pedagogical example and secondly



because it insures the connection with the usual anyon statistics. Then we turn to the  $d = 2$  case emphasizing the rôle of a finite  $\mathcal{W}$ -algebra. A short paragraph is also devoted to the generalization to higher dimensions. We conclude by an analysis which leads to recognize, in the  $d = 2$  more general framework, the anyonic statistics parameter. In what follows, we naturally use, as often as possible, the notations of [5].

### 3.1 Two particles in one dimension

The relative coordinate and momentum of the two particle system are denoted by:

$$x = x_{(1)} - x_{(2)} \quad p = \frac{1}{2} (p_{(1)} - p_{(2)}) \quad (3.1)$$

and satisfy the C.R.:

$$\{x, p\} = 1 \quad [x, p] = i \quad (3.2)$$

in either the classical or the quantum case.

The chosen observables:

$$E_+ = \frac{1}{2} p^2 \quad E_- = \frac{1}{2} x^2 \quad H = \frac{1}{4i} (xp + px) \quad (3.3)$$

close, in the quantum case, under the C.R.'s of the  $\mathcal{G} = sp(2, \mathbb{R}) \simeq sl(2, \mathbb{R})$ -like algebra:

$$[E_+, E_-] = 2H \quad [H, E_{\pm}] = \pm E_{\pm} \quad (3.4)$$

The realization (3.3), where as usual  $x^2$  acts as a multiplicative factor and  $p = -i\partial_x$ , is associated to the eigenvalue

$$\gamma_0 = -\frac{3}{16} \quad (3.5)$$

of the Casimir operator:

$$\Gamma = H^2 + \frac{1}{2} (E_+ E_- + E_- E_+) \quad (3.6)$$

Following [5], we wish to use a "coordinate" representation: then one natural observable to diagonalize is  $x^2$ .

It is certainly the most well known property that in  $sl(2, \mathbb{R})$ , one can diagonalize simultaneously one generator and the Casimir invariant, and also that these two generators constitute a complete set of commuting observables. However, let us remark that this property can be reobtained by considering the  $2 \times 2$  matrix representation of  $sl(2, \mathbb{R})$ :

$$J = \begin{pmatrix} J_0 & J_+ \\ J_- & -J_0 \end{pmatrix} \quad (3.7)$$

and determining the finite  $\mathcal{W}$ -algebra generated by  $\mathcal{G}^*$  quantities invariant under the subgroup  $G_+$  generated by  $E_+$ . Indeed, with a group transformation

$$J^g = g^{-1} J g \quad \text{with} \quad g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad (3.8)$$

we find in the highest weight basis

$$J^g = \begin{pmatrix} 0 & (J_+ J_- + J_0^2)/J_- \\ J_- & 0 \end{pmatrix} \quad (3.9)$$

Then, using the isomorphism  $i : \mathcal{G}^* \rightarrow \mathcal{G}$  as explained in section 2.3, one will recover the  $sl(2, \mathbb{R})$  Casimir operator as generating (besides  $E_-$  itself) the commutant of the  $E_-$ -generator. Of course, this approach reminds us the realization, in the affine case, of the  $W_2$  -or Virasoro- algebra via the Drinfeld-Sokolov construction, usually obtained with the constraint:  $J_- = 1$ .

Now, one can modify the  $sl(2, \mathbb{R})$  realization (3.3) into still a  $x$  and  $p$  one, but valid for any eigenvalue  $\gamma$  of  $\Gamma$ . Indeed, let us involve simultaneously the coordinate observable  $E_- = \frac{1}{2}x^2$  and the Casimir operator  $\Gamma$ . For such a purpose, we keep unchanged  $E_-$ , as well as the expression of  $H$ , which can be seen as a function of  $x^2$  and its conjugate  $\partial_{x^2}$ :

$$H = -(x^2 \partial_{x^2} + \frac{1}{4}) \quad (3.10)$$

Using (3.5) and (3.6), and considering the difference  $\Gamma - \Gamma_0$ , we deduce the shift on  $E_+$ :

$$E_+ \rightarrow E'_+ = E_+ + (\gamma + \frac{3}{16}) \frac{1}{x^2} \quad (3.11)$$

which does not affect the  $sl(2, \mathbb{R})$  C.R.'s.

As discussed by the authors of [5], the parameter:

$$\lambda = \gamma + \frac{3}{16} \quad (3.12)$$

can be directly related to the anyonic continuous parameter, with end point  $\lambda = 0$ , or  $\gamma = -\frac{3}{16}$ , corresponding to the boson and fermion cases.

## 3.2 Two particles in two dimensions

The algebra under consideration is generated by the quadratic homogeneous polynomials in the relative coordinates  $x_j$  and  $p_j$  ( $j = 1, 2$ ). One can recognize a realization of the  $sp(4, \mathbb{R})$  algebra, the generators of which can be conveniently separated into three subsets:

- the three (commuting) coordinate operators:

$$u = (x_1)^2 + (x_2)^2 \quad v = (x_1)^2 - (x_2)^2 \quad w = 2x_1x_2 \quad (3.13)$$

- the three (commuting) second order differential operators:

$$U = (p_1)^2 + (p_2)^2 \quad V = (p_1)^2 - (p_2)^2 \quad W = 2p_1p_2 \quad (3.14)$$

- the four first order differential operators, generating an  $sl(2, \mathbb{R}) \oplus gl(1)$  algebra:

$$\begin{aligned} C_s &= \frac{1}{4} \sum_{i=1}^2 (x_i p_i + p_i x_i) & C_d &= \frac{1}{4} (x_1 p_1 + p_1 x_1 - x_2 p_2 - p_2 x_2) \\ L &= x_1 p_2 - x_2 p_1 & M &= x_1 p_2 + x_2 p_1 \end{aligned} \quad (3.15)$$

$C_s$  being the Abelian factor.

Choosing the  $4 \times 4$  matrix representation of the  $\mathcal{G} = sp(4, \mathbb{R})$  algebra

$$M = m^{ij} \quad E_{ij} \quad i, j = 1, 2, 3, 4 \quad (3.16)$$

with  $m^{ij}$  real numbers satisfying

$$m^{ij} = (-1)^{i+j+1} m^{5-j,5-i} \quad (3.17)$$

the most general element of  $\mathcal{G}$  writes:

$$J = J_a t^a = \begin{bmatrix} -2J_{C_s} + J_M & J_L - 2J_{C_d} & -J_V & J_U - J_W \\ -J_L - 2J_{C_d} & -2J_{C_s} - J_M & -J_U - J_W & J_V \\ J_v & J_u + J_w & 2J_{C_s} + J_M & J_L - 2J_{C_d} \\ -J_u + J_w & -J_v & -J_L - 2J_{C_d} & 2J_{C_s} - J_M \end{bmatrix} \quad (3.18)$$

As in the  $d = 1$  case, the  $(x_j, p_j)$  realization as given in (3.13-3.15) is a particular one, associated with special eigenvalues of the  $sp(4, \mathbb{R})$  fundamental invariants  $\Delta^{(2)}$  and  $\Delta^{(4)}$ . But one can proceed as before in order to get a more general representation. Wishing to keep as observables the three commuting coordinate generators  $u, v, w$  (coordinate representation), one may think of determining in the  $\mathcal{G} = sp(4, \mathbb{R})$  enveloping algebra the commutant of these three generators. A basis for this subalgebra can be obtained, in a systematic way, via the technics of section 2. More precisely, such a commutant can be seen as a finite enlarged  $\mathcal{W}$ -algebra.

First, one remarks that the  $\mathcal{G}$  basis decomposition given in (3.13-3.15) naturally defines a  $\mathcal{G}$ -grading:

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \quad (3.19)$$

with  $\mathcal{G}_{-,+,0}$  generated by  $\{u, v, w\}$ ,  $\{U, V, W\}$  and  $\{C_s, C_d, L, M\}$  respectively. This grading is itself related to the  $sl(2, \mathbb{R})$  embedding obtained by taking the diagonal part of the  $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$  subalgebra of  $sp(4)$  [3, 9]. Note that the  $sl(2, \mathbb{R})_2$  negative root generator  $J_-$  can be seen on the matrix (3.18) by selecting the  $\mathcal{G}_-$  part, that is by replacing by zero all the entries except the ones where stand  $J_u, J_v$ , and  $J_w$ . To compare with the (usual) Hamiltonian reduction process, in this last framework the constraints could be chosen as:

$$J_u = J_v = 0, \quad J_w = 1 \quad (3.20)$$

There exist  $G_0$  elements connecting these two bases<sup>2</sup>: let  $T(J_u, J_v, J_w)$  one of these elements:

$$T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ J_v & J_u + J_w & 0 & 0 \\ -J_u + J_w & -J_v & 0 & 0 \end{pmatrix} \quad (3.21)$$

One knows that the decomposition of  $\mathcal{G}$  w.r.t.  $sl(2, \mathbb{R})_2$  representations is:

$$\mathcal{G} = D_0 \oplus 3D_1 \quad (3.22)$$

Using  $G_+$  transformations, it is just a question of computation to determine in the highest weight basis the four quantities (one for each  $sl(2, \mathbb{R})_2$  representation in  $\mathcal{G}$ ) which generate (besides  $u, v, w$  themselves) the set of invariants under this Borel subgroup. The highest weights can be determined as explicated in (2.19), or by acting with the  $T$  conjugation (3.21) on the highest weights relative to the  $sl(2)_2$  basis determined by (3.22). Practically, one has to find  $g_+ \in G_+$  such that:

$$g_+^{-1} J g_+ = J_u u + J_v v + J_w w + \sum_s \widetilde{W}_s \widetilde{l}^s \quad (3.23)$$

---

<sup>2</sup>We recall that  $\tilde{t}_- \neq 0$

keeping the notations of (3.13-3.22).

At the classical level, the  $\widehat{W}_s$  quantities Poisson commute with  $J_u, J_v$  and  $J_w$ . Using the  $i$ -transformation from  $\mathcal{G}^*$  to  $\mathcal{G}$  and the approach described in section 2.3, one can choose as generators of the commutant of the Abelian subalgebra  $\{u, v, w\}$  (in addition to  $u, v, w$  themselves):

$$\hat{S} = -uL + vM - 2wC_d \quad (3.24)$$

$$\begin{aligned} \hat{R} = \frac{-1}{24} \{ & -v^3W + 4v^2C_sL + 4v^2MC_d + v^2wV - vw^2W + \\ & + 2vwM^2 + vu^2W - 4uvMC_s - 4uvLC_d - 8vwC_d^2 + 8uwC_dC_s - wu^2V + \\ & + w^3V - 2uwLM + 4w^2LC_s - 4w^2C_dM + Sym. \} \end{aligned} \quad (3.25)$$

$$\begin{aligned} \hat{Q} = \frac{1}{120} [ & \{ 4u^2wLC_d - u^3wW + uw^3W + v^3uV - 2v^3ML + \\ & + v^2uwW + 4v^2wC_dL - 8uvwC_dM + v^4U + w^4U - 4v^2wMC_s + 2uv^2M^2 + \\ & + 2uv^2L^2 + 4w^3C_dL + 8uw^2C_d^2 + 2w^2uL^2 - 8vw^2C_dC_s - 2vu^2ML - 2w^2vML + \\ & + 2v^2w^2U - u^2v^2U - u^2w^2U - 8v^3C_sC_d - 4w^3C_sM - vu^3V + vuw^2V + \\ & + 8vu^2C_sC_d + 4u^2wC_sM + Sym. \} - \frac{28}{3}u(u^2 - v^2 - w^2) ] \end{aligned} \quad (3.26)$$

$$\hat{\mu} = \frac{1}{2} (vV + wW - 4C_s^2 + uU - 4C_d^2 - M^2 + L^2 + Sym.) + 8 \quad (3.27)$$

where  $Sym$  stands for the symmetrization of all the products appearing inside the parenthesis. These four quantities satisfy

$$\begin{aligned} [\hat{S}, \hat{Q}] &= -2i(u^2 - v^2 - w^2) \hat{R} \\ [\hat{S}, \hat{R}] &= 2i \hat{Q} \\ [\hat{Q}, \hat{R}] &= -8i(v^2 + w^2) ((u^2 - v^2 - w^2) \hat{\mu} - 2\hat{S}^2) \hat{S} \end{aligned} \quad (3.28)$$

We remark that, as conjectured in the general case, the r.h.s. in (3.28) are polynomials. When performing a "renormalization" of these four quantities:

$$S = \frac{1}{\sqrt{u^2 - v^2 - w^2}} \hat{S} \quad (3.29)$$

$$R = \frac{1}{\sqrt{u^2 - v^2 - w^2} \sqrt{v^2 + w^2}} \hat{R} \quad (3.30)$$

$$Q = \frac{1}{(u^2 - v^2 - w^2) \sqrt{v^2 + w^2}} \hat{Q} \quad (3.31)$$

$$\mu = \hat{\mu} \quad (3.32)$$

we recover the algebra obtained in [5]:

$$\begin{aligned} [S, Q] &= -2iR \\ [S, R] &= 2iQ \\ [Q, R] &= -8iS(\mu - 2S^2) \end{aligned} \quad (3.33)$$

One recognizes also the "deformed"  $gl(2)$  algebra, already considered in [3, 4].

Two remarks need to be added. First, we note that the generators of this  $gl(2)$ -like algebra as given in (3.23) are not all obtained as polynomials in the  $sp(4)$  generators.

As discussed in section 2.2 one can, in order to recover the polynomiality property of  $W$  generators obtained via Hamiltonian reduction in the highest weight gauge, consider the enlarged  $\mathcal{W}_S$  algebra by including the elements  $J_u, J_v, J_w$  forming the set  $\mathcal{G}_-^*$ . As already explained, the quotient  $\mathcal{W}_S/\mathcal{G}_-^*$  leads to a realization of the  $\mathcal{W}(sp(4), 2sl(2))$  algebra, whose quantization (section 2.3) provides the realization (3.29-3.32).

Secondly one can note that any state in a  $sp(4)$  representation is completely specified by six numbers, each eigenvalue of one of six simultaneously commuting operators: usually are considered the two  $sp(4)$  fundamental invariants  $\Delta^{(2)}$  and  $\Delta^{(4)}$ , then one can choose the two Casimir and two Cartan generators of the two  $sl(2)$  forming the maximal  $sl(2) \oplus sl(2)$  subalgebra in  $sp(4)$ . From the algebra (3.28),  $\mu, S$  and the  $gl(2)$ -like Casimir operator

$$\Delta^{Q,R,S} = Q^2 + R^2 + 4(\mu - 4)S^2 - 4S^4 \quad (3.34)$$

can obviously be diagonalized simultaneously, which, joined to the three generators  $u, v, w$  constitute a complete set of commuting observables as could be expected. Note also that the  $sp(4)$  fundamental invariants can easily be recognized in this basis via the relations [5]:

$$\Delta^{(2)} = \frac{1}{24}(\mu - 8) \quad ; \quad \Delta^{(4)} - \frac{1}{6}(\Delta^{(2)})^2 = \frac{1}{6912}(\Delta^{QRS} + 8\mu - 64) \quad (3.35)$$

Thus, finite  $\mathcal{W}$ -algebras can occasionally be used for the determination in a given simple Lie algebra of complete sets of commuting observables.

Now we are in position to generalize the  $(x_i, p_i)$  representation of  $sp(4)$  given in (3.13-3.15) to a more general one with the help of the  $S, Q, R, \mu$  generators. As in the  $sp(2)$  previous case, we wish to have a coordinate representation. So we will keep unchanged  $u, v, w$  as in (3.13). In order to follow closely the notations of [5] and facilitate the discussion, let us also introduce the parametrization:

$$u = r^2 \quad v = r^2 \sin \theta \cos 2\phi \quad w = r^2 \sin \theta \sin 2\phi \quad (3.36)$$

with  $0 \leq \theta \leq \frac{\pi}{2}$ ,  $0 \leq \phi \leq \pi$ , and also define:

$$\begin{aligned} C_\rho &= \cos(2\phi) \, 2C_d + \sin(2\phi) \, M \\ C_\phi &= -\sin(2\phi) \, 2C_d + \cos(2\phi) \, M \end{aligned} \quad (3.37)$$

The simplest quantity to examine is

$$\sqrt{u^2 - v^2 - w^2} \, S = -uL + \sqrt{v^2 + w^2} \, C_\phi. \quad (3.38)$$

We immediately note that for  $S = 0$ , one has the relation

$$uL = \sqrt{v^2 + w^2} \, C_\phi \quad (3.39)$$

in accordance with the  $(x_i, p_i)$  representation expressed in (3.13-3.15).

Let us decide to keep  $L$  unchanged. Then, for  $S \neq 0$ ,  $C_\phi$  becomes:

$$C_\phi \rightarrow C'_\phi = \frac{uL}{\sqrt{v^2 + w^2}} + \frac{\sqrt{u^2 - v^2 - w^2}}{\sqrt{v^2 + w^2}} \, S = -\frac{i}{\sin \theta} \partial_\phi + \cotg \theta \, S \quad (3.40)$$

After some reordering of the  $sp(4)$  generators via their CR's, we can express the quantities  $Q, R, \mu$ , denoted  $X_i$  ( $i = 1, 2, 3$ ) as follows:

$$X_i = f_i(u, v, w) \, U + g_i(u, v, w) \, V + h_i(u, v, w) \, W + k_i(u, v, w, L, C_s, C_\rho, C_\phi) \quad (3.41)$$

Using formula (3.40) for  $C_\phi$ , and keeping  $L, C_s, C_\rho$  unchanged (as well as of course  $u, v, w$ ), one directly gets the general coordinate realization of  $U, V, W$ , as given in [5] (eq. (75-78)). We just rewrite for illustration the  $U$  generator expression:

$$U = \frac{1}{r^2} \left[ -\partial_r r^2 \partial_r - \frac{4}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + (C_\phi)^2 + S^2 + \frac{\mu - 2S^2 + \sin \theta Q}{\cos^2 \theta} + \frac{3}{4} \right] \quad (3.42)$$

Thus the four operators associated with the finite  $\mathcal{W}$ -algebra are naturally used to obtain, after modification of the expressions of four  $sp(4)$  generators, the general "coordinate" realization of  $sp(4)$ . As developed in [5], matrix representations of the  $gl(2)$ -like algebra, in which the wave functions behave as vectors, have to be considered.

### 3.3 Two particles in $d > 2$

The above construction can directly be generalized to dimensions  $d > 2$ . There the quadratic homogeneous polynomials in  $x_i, p_i$  ( $i = 1 \dots d$ ) generate the symplectic algebra  $sp(2d, \mathbb{R})$ . One recognizes  $\frac{1}{2}d(d+1)$  coordinate generators  $x_i x_j$  ( $i, j = 1, \dots, d$ ),  $\frac{1}{2}d(d+1)$  ones relative to the  $p_i p_j$  second order differential operators, and  $d^2$  ones associated to the  $x_i p_j$ . Again one remarks the gradation:

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \quad (3.43)$$

with the Abelian subalgebras  $\mathcal{G}_{\pm 1}$  generated by the  $x_i x_j$  and  $p_i p_j$  quantities respectively, while  $\mathcal{G}_0$ , isomorphic to  $gl(d, \mathbb{R})$ , is spanned by the  $x_i p_j$ 's. Such a gradation is relative to the diagonal  $sl(2)$  in the regular subalgebra direct sum of  $d$   $sl(2)$  ones. The associated finite  $\mathcal{W}$ -algebra, will be of dimension  $d^2$  and a "deformed" version of the  $gl(d)$  algebra in the same way we have found in  $sp(4)$  an  $gl(2)$ -like algebra (3.28). More precisely, from the  $sl(2)$  decomposition

$$\mathcal{G} = \frac{d(d+1)}{2} D_1 \oplus \frac{d(d-1)}{2} D_0 \quad (3.44)$$

it is clear that the  $\mathcal{W}$ -algebra will contain a Lie subalgebra of dimension  $\frac{d(d-1)}{2}$ , and that the other  $\frac{d(d+1)}{2}$   $W$  generators will close with cubic (or lower degree) terms. When going from the bosonic representation to a general one, we will express the  $sp(2d)$  generators as functions of the  $x_i$ 's,  $p_i$ 's, and  $W$  generators. The set of observables will be constituted by the  $\frac{d(d+1)}{2}$   $x_i x_j$ 's and the  $d-1$  Casimir operators of the "deformed  $gl(d)$ " algebra, together with its  $d$  Cartan generators. We recover the right number of operators in order to form a complete set of commuting observables for  $sp(2d)$ . Note that the second order differential operators ( $\mathcal{G}_1$  part of  $sp(2d)$ ) exhaust the number of  $W$  generators only when  $d = 1$ , while in other cases, we will have to modify also  $\frac{d(d-1)}{2}$  zero grade generators.

### 3.4 Anyons in $d = 2$ ?

It is reasonable to expect the  $d = 2$  case to be a generalization of the  $d = 1$  one, in which the anyonic parameter is clearly identified. However, the author of [5] had a rather negative conclusion based on the following argument. They found the boson and fermion assigned to representations of dimension  $J = K + 1 = 1$  and 2 respectively of the  $\mathcal{W}$ -algebra (3.28). The parameter  $K$  is obviously discrete, which ruins an anyonic interpolation between bosons and fermions.

We would like to discuss the following alternative. First, we recall that the authors of [5] works with multivalued functions. We choose to work with univalued ones. Then, the operator  $L$  introduced in the previous section

$$L = -i \frac{\partial}{\partial \phi} \quad (3.45)$$

is not well-defined on the set  $L_2([0, \pi])$ . More precisely, for  $L$  to be self-adjoint, we have to:

- either restrict it on functions satisfying

$$\psi(0) = \lambda \psi(\pi) \quad (3.46)$$

where  $\lambda$  plays the rôle of an anyonic parameter ( $\lambda = 1$  characterizes the bosons, while  $\lambda = -1$  characterizes fermions)

- or modify the explicit form of  $L$

$$L = -i \frac{\partial}{\partial \phi} + \alpha \quad (3.47)$$

and apply it on functions such that

$$\psi(0) = \psi(\pi) \quad (3.48)$$

the anyonic parameters being here  $\alpha$  ( $\alpha = 0$  being the bosons, and  $\alpha = 1$  being the fermions) [12].

As we start with the bosonic realization, it is natural to assume that we are working with functions of kind (3.48). Then, the form (3.45), compared with (3.47) "confirms" that we are really working with bosons. However, this is true only for the  $x_i, p_j$  representation. When we study a more general representation, we have to choose which generators will be modified (see section 3.2). Until now, following [5], we had chosen to keep  $L$  unchanged. We present hereafter some scenarii that may help to recognize anyons in this framework.

(i) Let us choose to keep  $C_\phi$  unchanged, and transform  $L$ . Then, from (3.38), we get a new expression:

$$L = -i \frac{\partial}{\partial \phi} - S \cos \theta \quad (3.49)$$

By a computation analogous to the one performed in section 3.2, one can realize the other generators of  $sp(4, \mathbb{R})$  in accordance with (3.49). If we concentrate on the one-dimensional representations of the  $\mathcal{W}$ -algebra,  $S$  becomes a number which can be non-zero. Actually, it has been shown in [5] that the positivity of the two operators  $A_i = (p_i)^2 + (x_i)^2$  ( $i = 1, 2$ ) imposes  $S = \pm 2$ . Then, the real number  $\alpha = S \cos \theta$  could be interpreted as the anyonic parameter ( $\alpha = 0$  corresponding to the usual boson and fermion cases). Note however that the term  $\cos \theta$  is not constant on any  $s\ell(2)$  subalgebras built from  $L$ , since it does not commute with the other  $s\ell(2)$  generators. This means that we have to break the  $s\ell(2)$  symmetry to recognize anyons. For higher dimensional representations,  $S$  is a diagonal matrix, and we have a generalization of anyons directly related to the enlarged  $\mathcal{W}$ -algebra (we recall that  $\cos \theta = \sqrt{u^2 - v^2 - w^2} / u$ ).

(ii) One may think of not breaking the whole  $sp(4)$  symmetry. Then, one can choose for  $L$ :

$$L = -i \frac{\partial}{\partial \phi} - (r^2 \cos \theta)^2 S \quad (3.50)$$

The term  $(r^2 \cos \theta)^2$  commutes with the full  $sl(2)$  algebra generated by  $L, M, 2C_d$ . Indeed, setting

$$\rho_1 = r^2 \cos \theta \quad \text{and} \quad \rho_2 = r^2 \sin \theta \quad (3.51)$$

one can rewrite  $M$  and  $2C_d$  in the bosonic representation as

$$M = -2i \frac{r^2}{\rho_2} \left( \cos(2\phi) \rho_2 \frac{\partial}{\partial \rho_2} + \sin(2\phi) \frac{\partial}{\partial(2\phi)} \right) \quad (3.52)$$

$$2C_d = -2i \frac{r^2}{\rho_2} \left( \sin(2\phi) \rho_2 \frac{\partial}{\partial \rho_2} - \cos(2\phi) \frac{\partial}{\partial(2\phi)} \right) \quad (3.53)$$

$$\text{with } r^2 = \sqrt{\rho_1^2 + \rho_2^2} \quad (3.54)$$

which clearly shows that  $r^2 \cos \theta \equiv \rho_1$  commutes with the whole  $sl(2, \mathbb{R})$  subalgebra, and therefore is constant on each  $sl(2)$  representation.

In a finite dimensional  $sl(2)$  representation,  $L$  has only (half-)integer eigenvalues. In order to get a continuous parameter, we have to consider infinite dimensional  $sl(2)$ -representations. Then  $\rho_1$  could be seen as a continuous free parameter, and  $\rho_1 S$  interpreted as a generalized anyonic parameter. In this context, focussing on the one-dimensional representation of the  $\mathcal{W}$ -algebra, a given  $sl(2)$  representation would define one anyon type. In the complete  $sp(4)$  representation we recognize all the anyons, two anyonic  $sl(2)$  representations being related by a  $sp(4)$  transformation. Such a situation generalizes to higher dimensional  $\mathcal{W}$ -representations.

(iii) Finally, one chooses

$$L = -i \frac{\partial}{\partial \phi} - S \quad (3.55)$$

We still have to take infinite dimensional representations for the  $sl(2)$  subalgebra. Considering the subalgebra generated by  $L, V + v, W + w$ , it is enough to release the positivity of the spectrum for  $A_1 - A_2$  (i.e. ask the spectrum of  $A_1 + A_2$  only to be positive instead of the spectrum of both  $A_1$  and  $A_2$ ) to allow infinite dimensional representations.

## 4 Conclusion

We have presented a construction of finite  $\mathcal{W}$ -algebras in which  $W$  elements are functions of all the current components  $J_a$ , in opposition to the usual case where some of these quantities are assigned to constants. Such an approach has been developed when the associated symmetry subgroup  $H$ , w.r.t. which are determined the  $J$  invariant quantities, is Abelian. Our construction can be generalized for a more general  $H$ , or in other words for any  $sl(2)$  embeddings in  $\mathcal{G}$ . The extension to the affine case can also be considered. These works are in progress and will be presented elsewhere [13].

Then, we have recognized a class of such finite  $\mathcal{W}$ -algebras in the framework of Heisenberg quantization for  $N = 2$  identical particles. From the explicit expressions of the  $W$  generators, one can deduce in  $d$  dimensions a general realization of the observable algebra



under consideration. Moreover, the anyonic parameter being directly related to the unique  $W$  generator in the  $d = 1$  case, we have tried to identify it again in the larger  $\mathcal{W}$ -algebra relative to the  $d = 2$  case. Interpretation of the whole algebra still needs to be precised. Of course, the case of  $N > 2$  identical particles, in which the algebra of observables becomes infinite [5] deserves a careful study and retains now our attention.

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